A CORRIGIBLE HORIZONTAL GYROCOMPASS

(KORREKTIRUEMYI GIROGORIZONTKOMPAS)

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A horizontal gyrocompass which has a sensing element on a stabilized horisontal platform aligning itself along a meridian by means of a correcting device, may be called a corrigible horizontal gyrocompass.

Here we investigate one of the possible designs of a corrigible horisontal gyrocompass whose sensing element is a gyrosphere of a conventional two-rotor gyrocompass; the center of gravity of the gyrosphere coincides with its geometric center. The correcting device produces the turning moments [1] on the corresponding axes of the gyrosphere and on the axis of the casing of one of the gyroscopes inside the gyrosphere.

1. The equations of motion of a gyrosphere [2 and 3] whose center of gravity coincides with its geometric center in the geographical system of reference have the following form (1.1)

$$[2B\cos(\varepsilon-\delta)\sin\beta]' + 2B\cos(\varepsilon-\delta)(u_1\cos\alpha\cos\beta + u_2\sin\alpha\cos\beta) = M_\zeta$$

$$2B\cos(\varepsilon-\delta)(\alpha'\cos\beta + u_1\sin\alpha\sin\beta - u_2\cos\alpha\sin\beta + u_3\cos\beta) = M_{x^*}$$

$$[2B\cos(\varepsilon-\delta)]' = M_z$$

$$2B\sin(\varepsilon-\delta)(\alpha'\sin\beta + \gamma' - u_1\sin\alpha\cos\beta + u_2\cos\alpha\cos\beta + u_3\sin\beta) = $$
$$= \varkappa\sin\delta\cos\delta - M_{y_1}$$

$$\left(u_1 = -\frac{v_N}{R},\quad u_2 = U\cos\varphi + \frac{v_E}{R},\quad u_3 = U\sin\varphi + \frac{v_E}{R}\tan\varphi\right)$$

Equations (1.1) describe the motion of a gyrosphere with respect to the $\xi\eta\zeta$ coordinate system (Fig.1), whose origin is in the geometric center of the gyrosphere, the ζ-axis is along the Earth's radius, and the ξ and η axes are horizontal and pointing, respectively, to the east and to the north. Let α, β and γ be the Eulerian angles, determinung the orientation of the gyrosphere with respect to the $\xi\eta\zeta$ axes. Besides, the angle α is the angle of rotation of the gyrosphere about the ζ-axis, β is the angle of rotation about the negative section of the line of nodes x^*, and γ is the angle of rotation of the gyrosphere about its z-axis.

The angle between the spin axes of the gyroscopes inside the gyrosphere equals $2(\varepsilon-\delta)$. At equilibrium, when the rotors do not spin, the angle between their axes is 2ε, and δ is the angle of rotation of one of the gyroscopes about y_1 which is the axis of its casing. Since the casings of both gyroscopes are connected by an antiparallelogram, the angle of rotation of the casing of the second gyroscope equals $-\delta$. The middle link of the antiparallelogram is connected with the internal surface of the gyrosphere by springs of rigidity $\varkappa$.

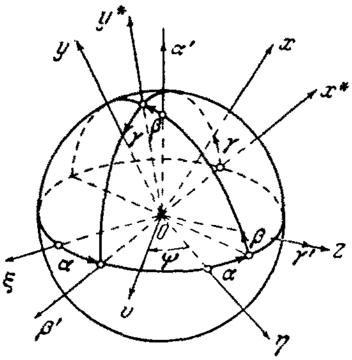

Fig. 1

In Equations (1.1) the angular momentum of each of the two identical gyroscopes inside the gyrosphere is B , and the components of the instantaneous angular velocity of the coordinate tetragon $\xi\eta\zeta$ are u_1 , u_2 , u_3 . The northern and the eastern components of ship's velocity with respect to the Earth's sphere are v_N and v_E. The angular velocity of the Earth's rotation is U , the latitude of the ship is φ , the Earth's radius is R . The moments of the exterior forces applied to the system are M_ζ, M_{x^*}, M_z and M_{y_1}

Since the instrument is on a stabilized horizontal platform the angles β and γ which the z- and the x-axes of the gyrosphere (Fig.1) make with the horizontal plane are measurable.

The angle between the horizontal projection of the z-axis of the gyrosphere and the ship's velocity vector $\mathbf{v}$ (Fig.1) with respect to the Earth's sphere equals $\psi + \alpha$, where ψ is the course of the ship. Since the angle $\psi + \alpha$ is measurable, therefore if we have an instrument determining the velocity of the ship with respect to the Earth's sphere we can determine the velocity components of the ship $v \cos(\psi + \alpha)$ and $v \sin(\psi + \alpha)$. Knowing the latitude of the ship φ , we can apply to the gyrosphere the correcting turning moments

$$M_\zeta = -2B\cos\varepsilon\frac{v}{R}\cos(\psi+\alpha)\cos\beta - \mu K\sin\beta,\quad M_z = -K\sin\gamma$$

$$M_{x^*} = -2B\cos\varepsilon\frac{v}{R}\sin(\psi+\alpha)\sin\beta + \tag{1.2}$$

$$+ 2B\cos\varepsilon\left[U\sin\varphi + \frac{v}{R}\sin(\psi+\alpha)\tan\varphi\right]\cos\beta + K\sin\beta$$

The correcting turning moment about y_1 which is the axis of the gyroscope's casing is assumed to be

$$M_{y_1} = -2B\sin\varepsilon\left[U\cos\varphi + \frac{v}{R}\sin(\psi+\alpha)\right]\cos\beta - \tag{1.3}$$

$$-2B\sin\varepsilon\left[U\sin\varphi + \frac{v}{R}\sin(\psi+\alpha)\tan\varphi\right]\sin\beta + \sigma K\sin\gamma$$

Here μ , K and σ are constant coefficients. It is natural to assume that the lattitude of the ship $\varphi < 90°$.

Taking into account that (1.4)

$$v\cos(\psi+\alpha) = v_N\cos\alpha - v_E\sin\alpha,\qquad v\sin(\psi+\alpha) = v_E\cos\alpha + v_N\sin\alpha$$

$$(v_N = v\cos\psi,\ v_E = v\sin\psi)$$

and substituting (1.2) into (1.3) we shall reduce Equations (1.1) to

$$(H^*\sin\beta)' + (H^* - H)\left(-\frac{v_N}{R}\cos\alpha\cos\beta + \frac{v_E}{R}\sin\alpha\cos\beta\right) +$$

$$+ H^*U\cos\varphi\sin\alpha\cos\beta + \mu K\sin\beta = 0$$

$$H^*\alpha'\cos\beta - H^*U\cos\varphi\cos\alpha\sin\beta +$$
$$-(H^*-H)\left(-\frac{v_N}{R}\sin\alpha\sin\beta - \frac{v_E}{R}\cos\alpha\sin\beta + U\sin\varphi\cos\beta\right) + H^*\frac{v_E}{R}\tan\varphi\cos\beta$$
$$-H\frac{v_E}{R}\tan\varphi\cos\alpha\cos\beta - H\frac{v_N}{R}\tan\varphi\sin\alpha\cos\beta - K\sin\beta = 0$$
$$H^{*\prime} + K\sin\gamma = 0$$
$$\Xi^*(\alpha'\sin\beta + \gamma') - \varkappa\sin\delta\cos\delta +$$
$$+(\Xi^* - \Xi)\left(\frac{v_N}{R}\sin\alpha\cos\beta + \frac{v_E}{R}\cos\alpha\cos\beta + U\sin\varphi\sin\beta\right) +$$
$$+\Xi^*U\cos\varphi\cos\alpha\cos\beta - \Xi U\cos\varphi\cos\beta + \Xi^*\frac{v_E}{R}\tan\varphi\sin\beta -$$
$$-\Xi\frac{v_E}{R}\tan\varphi\cos\alpha\sin\beta - \Xi\frac{v_N}{R}\tan\varphi\sin\alpha\sin\beta + \sigma K\sin\gamma = 0 \qquad (1.5)$$

$$(H^* = 2B\cos(\varepsilon - \delta), \quad H = 2B\cos\varepsilon, \quad \Xi^* = 2B\sin(\varepsilon - \delta), \quad \Xi = 2B\sin\varepsilon)$$

It is easily seen that the system of differential equations (1.5) has the particular solution

$$\alpha = \beta = \gamma = \delta = 0 \qquad (1.6)$$

In this way, even at a most arbitrary motion of the ship $\mathbf{v} = \mathbf{v}(t)$ the equilibrium direction of the z-axis of the gyrosphere is north. This means that the velocity deviation of a corrigible gyrocompass equals zero.

A gyrocompass works satisfactorily only when its oscillations about its positions of equilibrium $\alpha = \beta = \gamma = \delta = 0$ are damped. The equations for the variations, are obtained from (1.5) by assuming that the angles α, β, γ and δ are small and have the form

$$\alpha' - \frac{v_N}{R}\tan\varphi\,\alpha - \left(\frac{K}{H} + U\cos\varphi\right)\beta + \frac{\Xi}{H}\left(U\sin\varphi + \frac{v_E}{R}\tan\varphi\right)\delta = 0$$
$$\beta' + \frac{\mu K}{H}\beta + U\cos\varphi\,\alpha - \frac{\Xi}{H}\frac{v_N}{R}\delta = 0 \qquad (1.7)$$
$$\gamma' + \frac{\sigma K}{\Xi}\gamma - \frac{1}{\Xi}\left[\varkappa + H\left(U\cos\varphi + \frac{v_E}{R}\right)\right]\delta = 0, \quad \delta' + \frac{K}{\Xi}\gamma = 0$$

2. Since $v_N = v_N(t)$, $v_E = v_E(t)$, $\varphi = \varphi(t)$, then Equations (1.7) form a system of linear differential equations with variable coefficients.

The sufficient conditions of an asymptotic stability of the particular solution (1.6) which determines the position of equilibrium (more properly the steady motion) of a gyrocompass we find from the Liapunov function, which can be constructed by the method presented in [4]. Denoting

$$f_1(t) = \frac{v_N}{R}\tan\varphi, \quad f_2(t) = U\cos\varphi, \quad F_1(t) = \frac{\Xi}{H}\left(U\sin\varphi + \frac{v_E}{R}\tan\varphi\right)$$
$$F_2(t) = \frac{\Xi}{H}\frac{v_N}{R}, \quad F_3(t) = \frac{H}{\Xi}\left(U\cos\varphi + \frac{v_E}{R}\right) \qquad (2.1)$$

we shall reduce Equations (1.7) to the form

$$\alpha' - \frac{K}{H}\beta = f_1(t)\alpha + f_2(t)\beta - F_1(t)\delta$$

$$\beta' + \frac{\mu K}{H}\beta + s\alpha = [s - f_2(t)]\alpha + F_2(t)\delta$$

$$\gamma' + \frac{\sigma K}{\Xi}\gamma - \frac{\varkappa}{\Xi}\delta = F_3(t)\delta, \qquad \delta' + \frac{K}{\Xi}\gamma = 0 \tag{2.2}$$

We shall select the coefficient s in the second equation of (2.2) in such a way that the system of differential equations

$$\alpha' - \frac{K}{H}\beta = 0, \qquad \beta' + \frac{\mu K}{H}\beta + s\alpha = 0 \tag{2.3}$$

will have the characteristic equation

$$\lambda^2 + \frac{\mu K}{H}\lambda + \frac{sK}{H} = 0 \tag{2.4}$$

with the pair of complex roots

$$\lambda_1, \lambda_2 = \varepsilon_1 \pm i\omega_1, \quad \varepsilon_1 = -\frac{\mu K}{2H}, \qquad \omega_1 = \left(\frac{sK}{H} - \varepsilon_1^2\right)^{1/2} \tag{2.5}$$

In this case the coefficient s should satisfy the inequality

$$s > \mu^2 K / 4H \tag{2.6}$$

Let us mention that the coefficient s is not a parameter of the system, but its selection [4] determines the subsequent transformation from the old to the new variables and parameters of the Liapunov function, which result from this transformation.

We shall select the parameters $\varkappa$, σ and r to satisfy the condition

$$\varkappa > {}^1/_4\sigma^2 K \tag{2.7}$$

Besides, the differential equations

$$\gamma' + \frac{\sigma K}{\Xi}\gamma - \frac{\varkappa}{\Xi}\delta = 0, \qquad \delta' + \frac{K}{\Xi}\gamma = 0 \tag{2.8}$$

will have the characteristic equation

$$\Lambda^2 + \frac{\sigma K}{\Xi}\Lambda + \frac{\varkappa K}{\Xi^2} = 0 \tag{2.9}$$

with the pair of complex roots

$$\Lambda_1, \Lambda_2 = \varepsilon_2 \pm \omega_2 i \quad \left(\varepsilon_2 = -\frac{\sigma K}{2\Xi}, \quad \omega_2 = \left(\frac{\varkappa K}{\Xi^2} - \varepsilon_2^2\right)^{1/2}\right) \tag{2.10}$$

Let us introduce the new variable $x_1, \ldots, x_4$ through

$$\alpha = x_1, \quad \beta = \frac{\varepsilon_1 H}{K}x_1 + \frac{\omega_1 H}{K}x_2, \quad \gamma = \omega_2 x_3 + \varepsilon_2 x_4, \quad \delta = -\frac{K}{\Xi}x_4 \tag{2.11}$$

From (2.11) follows that

$$x_1 = \alpha, \quad x_2 = -\frac{\varepsilon_1}{\omega_1}\alpha + \frac{K}{\omega_1 H}\beta, \quad x_3 = \frac{1}{\omega_2}\gamma + \frac{\varepsilon_2 \Xi}{\omega_2 K}\delta, \quad x_4 = -\frac{\Xi}{K}\delta \tag{2.12}$$

By (2.2) the new variables $x_1, \ldots, x_4$, will satisfy the following system of differential equations:

$$x_1' = \left[\varepsilon_1 + f_1(t) + \frac{\varepsilon_1 H}{K} f_2(t)\right] x_1 + \omega_1 \left[1 + \frac{H}{K} f_2(t)\right] x_2 + \frac{K}{\Xi} F_1(t)\, x_4$$

$$x_2' = -\left\{\omega_1 - \frac{K}{\omega_1 H}[s - f_2(t)] + \frac{\varepsilon_1}{\omega_1}\left[f_1(t) + \frac{\varepsilon_1 H}{K} f_2(t)\right]\right\} x_1 + \\ + \varepsilon_1 \left[1 - \frac{H}{K} f_2(t)\right] x_2 - \frac{K}{\Xi}\left[\frac{\varepsilon_1}{\omega_1} F_1(t) + \frac{K}{\omega_1 H} F_2(t)\right] x_4$$

$$x_3' = \varepsilon_2 x_3 - \left[\omega_2 + \frac{K}{\omega_2 \Xi} F_3(t)\right] x_4, \qquad x_4' = \omega_2 x_3 + \varepsilon_2 x_4 \qquad (2.13)$$

The Liapunov function is the negative-definite function

$$V = -{}^1/_2 (x_1^2 + x_2^2 + p x_3^2 + p x_4^2) \qquad (p > 0) \qquad (2.14)$$

By (2.13) its time derivative, has the form

$$V' = a_{11} x_1^2 + 2a_{12} x_1 x_2 + 2a_{13} x_1 x_3 + 2a_{14} x_1 x_4 + a_{22} x_2^2 + 2a_{23} x_2 x_3 + 2a_{24} x_2 x_4 + \\ + a_{33} x_3^2 + 2a_{34} x_3 x_4 + a_{44} x_4^2 \qquad (2.15)$$

where

$$a_{11} = -\left[\varepsilon_1 + f_1(t) + \frac{\varepsilon_1 H}{K} f_2(t)\right]$$

$$a_{12} = \frac{1}{2}\left\{\frac{\varepsilon_1}{\omega_1} f_1(t) - \frac{H}{\omega_1 K}(\omega_1^2 - \varepsilon_1^2) f_2(t) - \frac{K}{\omega_1 H}[s - f_2(t)]\right\}$$

$$a_{13} = 0, \quad a_{14} = -\frac{K}{2\Xi} F_1(t), \quad a_{22} = -\varepsilon_1\left[1 - \frac{H}{K} f_2(t)\right]$$

$$a_{23} = 0, \quad a_{24} = \frac{K}{2\Xi}\left[\frac{\varepsilon_1}{\omega_1} F_1(t) + \frac{K}{\omega_1 H} F_2(t)\right]$$

$$a_{33} = p a_{33}^*, \quad a_{34} = p a_{34}^*, \quad a_{44} = p a_{44}^*, \quad a_{33}^* = -\varepsilon_2$$

$$a_{34}^* = \frac{!K}{2\omega_2 \Xi} F_3(t), \quad a_{44}^* = -\varepsilon_2 \qquad (2.16)$$

By the theorem of Sylvester, the quadratic form (2.15) is positive-definite if at any instant of time t the principal diagonal minors of its discriminant

$$D = \begin{vmatrix} a_{11} & a_{12} & 0 & a_{14} \\ a_{12} & a_{22} & 0 & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ a_{14} & a_{24} & a_{34} & a_{44} \end{vmatrix} \qquad (2.17)$$

are positive.

The conditions for having the principal diagonal minors of the discriminant (2.17) positive is

$$a_{11} > 0, \quad \Delta = a_{11} a_{22} - a_{12}^2 > 0, \quad a_{33}^* > 0$$

$$p\Delta^* \Delta + a_{33}^* (2a_{12} a_{14} a_{24} - a_{14}^2 a_{22} - a_{24}^2 a_{11}) > 0 \qquad (\Delta^* = a_{33}^* a_{44}^* - a_{34}^{*2}) \qquad (2.18)$$

It is easily seen that if the conditions

$$a_{11} > 0, \quad \Delta > 0, \quad a_{33}^* > 0, \quad \Delta^* > 0 \qquad (2.19)$$

are satisfied, then the last condition in (2.18) can be satisfied by choosing the coefficient p sufficiently large. In Expression (2.15) the coefficient p is not fixed.

Thus, the inequalities (2.19) can be regarded as conditions which, if satisfied, make the principal diagonal minors of the discriminant positive.

The inequalities (2.19) should be satisfied at any instant of time t and represent the sufficient conditions of the asymptotic stability of the equilibrium position (1.6) of a horizontal gyrocompass.

3. If in the initial equations (1.1) we set $\varepsilon = 0$, $\gamma \equiv 0$, $\delta \equiv 0$ and if we also assume that in (1.2) and (1.3) the correcting moments are $M_z \equiv 0$, $M_{y_1} \equiv 0$, then we shall get the equations of motion of a corrigible gyrocompass with one rotor, investigated previously in [5].

With the correcting moments M_ζ and M_{x*}, determined respectively by (1.2), the position of equilibrium of a corrigible gyrocompass with one rotor [5] is at

$$\alpha = 0, \qquad \beta = 0 \qquad (3.1)$$

which means that in this instrument, as in the previously discussed horizontal gyrocompass, the velocity deviation is zero.

The equations for variations with respect to the position of equilibrium (3.1) for a gyrocompass with one rotor are [5]

$$\alpha' - \frac{v_N}{R} \tan\varphi\, \alpha - \left(\frac{K}{H} + U\cos\varphi\right)\beta = 0, \qquad \beta' + \frac{\mu K}{H}\beta + U\cos\varphi\,\alpha = 0 \qquad (3.2)$$

and can be obtained from (1.7) if we substitute in these equations $\gamma \equiv 0$, $\delta \equiv 0$.

The sufficient conditions of stability at the position of equilibrium (3.1) of a gyrocompass with one rotor [5]

$$a_{11}(t) > 0, \qquad a_{11}(t)\, a_{22}(t) - [a_{12}(t)]^2 > 0 \qquad (3.3)$$

where a_{11}, a_{12}, a_{22} are determined by (2.17), follow directly from the obtained previously sufficient conditions of stability (2.19).

4. As an example we shall consider a horizontal gyrocompass whose parameters are

$$\frac{K}{H} = 3.6\,\mathrm{sec}^{-1}, \quad \mu = 0.005, \quad \frac{H}{\Xi} = 1, \quad \frac{\varkappa}{\Xi} = 0.0016\,\mathrm{sec}^{-1}, \quad \sigma = 0.04$$

The coefficient s, which determines the transformation (2.2) and the parameters (2.6) in the Liapunov function is assumed to be $s = 4\cdot 10^{-5}\,\mathrm{sec}^{-1}$.

With these parameters the sufficient condition of stability (2.20) is satisfied at all points inside the rectangular parallelepiped

$$|\varphi| \leqslant \varphi_m, \quad |v_N| \leqslant v_{N_m}, \quad |v_E| \leqslant v_{E_m}$$

Here

$$\varphi_m = 85^\circ, \quad v_{N_m} = 600\,\mathrm{m\ sec}^{-1}, \quad v_{E_m} = 600\,\mathrm{m\ sec}^{-1} \qquad (4.1)$$

Consequently, this corrigible horizontal gyrocompass can be used in aviation, as described in [1].

Let us mention that since $\varphi' = v_N/R$, the latitude of the ship is

$$\varphi(t) = \varphi(0) + \int_0^t \frac{v_N(\tau)}{R} d\tau \qquad (4.2)$$

In this way the position of equilibrium of a horizontal gyrocompass preserves the asymptotic stability even if the compass moves arbitrarily with velocity function $\mathbf{v} = \mathbf{v}(t)$, on the condition that $v_N(t)$, $v_E(t)$ and $\varphi(t)$ are inside the region (4.1).

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